



TITLE:

# Generalized Poisson integrals on unbounded domains and their applications

AUTHOR(S):

Yoshida, Hidenobu; Miyamoto, Ikuko

---

CITATION:

Yoshida, Hidenobu ...[et al]. Generalized Poisson integrals on unbounded domains and their applications. 数理解析研究所講究録 1994, 890: 31-62

ISSUE DATE:

1994-12

URL:

<http://hdl.handle.net/2433/84375>

RIGHT:

# Generalized Poisson integrals on unbounded domains and their applications

千葉大理 吉田 英信 (Hidenobu Yoshida)

千葉大理 宮本 育子 (Ikuko Miyamoto)

## PART 1. Introduction and the case of half-spaces

Let  $\mathbb{R}$  and  $\mathbb{R}_+$  be the set of all real numbers and all positive real numbers, respectively. The boundary and the closure of a set  $S$  in the  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  ( $n \geq 2$ ) are denoted by  $\partial S$  and  $\bar{S}$ , respectively. We also introduce the spherical coordinates  $(r, \theta)$ ,  $\theta = (\theta_1, \theta_2, \dots, \theta_{n-1})$ , in  $\mathbb{R}^n$  which are related to the cartesian coordinates  $(X, y)$ ,  $X = (x_1, x_2, \dots, x_{n-1})$  by the formulas

$$x_1 = r \left( \prod_{j=1}^{n-1} \sin \theta_j \right) \quad (n \geq 2), \quad y = r \cos \theta_1,$$

and if  $n \geq 3$ ,

$$x_{n+1-k} = r \left( \prod_{j=1}^{k-1} \sin \theta_j \right) \cos \theta_k \quad (2 \leq k \leq n-1),$$

where

$$0 \leq r < +\infty, \quad 0 \leq \theta_j \leq \pi \quad (1 \leq j \leq n-2; n \geq 3), \quad -2^{-1}\pi < \theta_{n-1} \leq 2^{-1}3\pi \quad (n \geq 2).$$

The unit sphere (the unit circle, if  $n=2$ ) and the upper half unit sphere  $\{(1, \theta_1, \theta_2, \dots, \theta_{n-1}) \in \mathbb{R}^n; 0 \leq \theta_1 < \frac{\pi}{2}\}$  (the upper half unit circle  $\{(1, \theta_1) \in \mathbb{R}^2; -\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}\}$ , if  $n=2$ ) in  $\mathbb{R}^n$  are denoted by  $S^{n-1}$  and  $S_+^{n-1}$ , respectively. The half-space

$$\{(X, y) \in \mathbb{R}^n; X \in \mathbb{R}^{n-1}, y > 0\} = \{(r, \theta) \in \mathbb{R}^n; \theta \in S_+^{n-1}, 0 < r < +\infty\}$$

is denoted by  $T_n$ .

Given a domain  $D \subset \mathbb{R}^n$  and a continuous function  $g$  on a subset  $S \subset \partial D$ , we say that  $h$  is a solution of the (classical) Dirichlet problem on  $D$  with  $g$ , if  $h$  is harmonic in  $D$  and

$$\lim_{P \in D, P \rightarrow Q} h(P) = g(Q)$$

for every  $Q \in S$ . If  $D$  is a bounded domain and  $g$  is a bounded function on  $\partial D$ , then the existence of a solution of the Dirichlet problem and its uniqueness is completely known (e.g. see [10, Theorem 5.21]). Otherwise we may suppose that  $D$  is always an unbounded domain by using the Kelvin transformation.

When  $D$  is the typical unbounded domain  $T_n$ , the following results are known. Let  $g(X)$  be a continuous function on  $\partial T_n = \mathbb{R}^{n-1}$  satisfying (1) with a non-negative integer  $\ell$ :

$$(1) \quad \int_{\mathbb{R}^{n-1}} \frac{|g(X)|}{1+|X|^{n+\ell}} dX < +\infty.$$

Then Armitage [1, Theorem 2] gave the explicit form of a solution of the Dirichlet problem on  $T_n$  with  $g$  (also see Siegel [15, P.1 and p.7]). Further, for any continuous function  $g(X)$  on  $\partial T_n$  Finkelstein and Scheinberg [7] showed the existence of a solution of the Dirichlet problem on  $T_n$  with  $g$  and Gardiner [8] gave the solution explicitly.

About the uniqueness of solutions of the Dirichlet problem on  $T_n$ , Helms [11, p.42 and p.158] states that even if  $g(X)$  is a bounded continuous function on  $\partial T_n$ , the solution of the Dirichlet problem on  $T_n$  with  $g$  is not unique and to obtain the unique solution  $H(P)$  ( $P=(X,y) \in T_n$ ) we must specify the behavior of  $H(P)$  as  $y \rightarrow +\infty$ . In connection with this remark, Siegel [15, Theorems 1 and 3] gave the following result to more restricted boundary function  $g$  than (1).

Let  $\ell$  be a non-negative integer. If  $g(X)$  ( $X \in \partial T_n = \mathbb{R}^{n-1}$ ) is a continuous function on  $\partial T_n$  such that

$$|g(X)| \leq G(X) \quad (X \in \mathbb{R}^{n-1}, |X|=x>0)$$

for a continuous function  $G(x)$  ( $x \in \mathbb{R}$ ),  $G(x)=G(-x)$ ,

$$\int_{-\infty}^{+\infty} \frac{|G(x)|}{1+|x|^{n+\ell}} dx < +\infty,$$

then there exists a solution  $H(T_n, \ell; g)(P)$  of the Dirichlet problem on  $T_n$  with  $g$  satisfying

$$H(T_n, \ell; g)(P) = o(r^{\ell+1}/\cos \theta_1) \quad (r \rightarrow +\infty) \\ (P=(r, \theta) \in T_n, \theta=(\theta_1, \theta_2, \dots, \theta_{n-1})).$$

If  $h(P)$  is a solution of the Dirichlet problem on  $T_n$  with  $g$  such that

$$h(P) = o(r^{\ell+1}/\cos \theta_1) \quad (P=(r, \theta) \in T_n),$$

then

$$h(P) = H(T_n, \ell; g)(P) + \Gamma(h)(P) \quad (P \in T_n),$$

where  $\Gamma(h)(P)$  is a harmonic polynomial (of  $P=(x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ ) of degree  $\ell$  vanishing on

$$\partial T_n = \{(x_1, x_2, \dots, x_{n-1}, 0) \in \mathbb{R}^n; (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

To answer the question of Siegel [15, p.8] Yoshida [19] proved

**THEOREM  $Y_1$**  [19, Theorems 1 and 2]. Let  $g(Q)$  be a continuous function on  $\partial T_n$  ( $n \geq 2$ ) satisfying (1) with a non-negative integer  $\ell$ . Then there exists a solution  $H(T_n, g; \ell)(P)$  of the Dirichlet problem with  $g$  satisfying

$$\lim_{r \rightarrow \infty} r^{-\ell-1} \int_{S_+^{n-1}} H(T_n, \ell; g)(r, \theta) \cos \theta_1 d\sigma_\theta = 0$$

$$(P=(r, \theta) \in T_n, \theta=(\theta_1, \theta_2, \dots, \theta_{n-1})),$$

where  $d\sigma_\theta$  is the surface element of  $S^{n-1}$ . If  $h(P)$  is a solution of

the Dirichlet problem on  $T_n$  with  $g$  satisfying

$$\lim_{r \rightarrow \infty} r^{-\ell-1} \int_{S_+^{n-1}} h^+(r, \theta) \cos \theta_1 d\sigma_\theta = 0,$$

then

$$h(P) = H(T_n, \ell; g)(P) + \pi(h)(P), \quad \pi(h)(P) = \begin{cases} y\pi^*(h)(P) & (\ell \geq 1) \\ 0 & (\ell = 0) \end{cases}$$

for every  $P = (X, y) \in T_n$ , where  $\pi^*(h)(P)$  is a polynomial of

$P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$  of degree at most  $\ell-1$  and even with respect to the variable  $y$ .

## PART 2. The conical case

### 1. Introduction

A half-space is a special one of more general unbounded domains cones. To generalize Theorem  $Y_1$  to results about cones, we shall first pay attention to Yoshida's result [18, Theorem 3] concerning the Dirichlet problem on a cone. To state it, we need some preliminaries.

Let  $\Delta_n$  ( $n \geq 2$ ) be the Laplace operator and  $\Lambda_n$  be the spherical part of the spherical coordinates of  $\Delta_n$ :

$$\Delta_n = \frac{n-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \Lambda_n.$$

Given a domain  $\Omega$  on  $S^{n-1}$  ( $n \geq 2$ ), consider the Dirichlet problem for

$$(2) \quad \begin{aligned} (\Delta_n + \lambda)F &= 0 && \text{on } \Omega \\ F &= 0 && \text{on } \partial\Omega. \end{aligned}$$

we denote the least positive eigenvalue of (2) by  $\lambda(\Omega, 1)$  and the normalized positive eigenfunction corresponding  $\lambda(\Omega, 1)$  by  $f_1^\Omega(\theta)$ . We shall denote two solutions of the equation

$$t^2 + (n-2)t - \lambda(\Omega, 1) = 0$$

by  $\alpha(\Omega, 1)$ ,  $-\beta(\Omega, 1)$  ( $\alpha(\Omega, 1)$ ,  $\beta(\Omega, 1) > 0$ ). Given a domain  $\Omega$  on  $S^{n-1}$ , the set

$$\{(r, \theta) \in \mathbb{R}^n; (1, \theta) \in \Omega, r \in \mathbb{R}_+\} \quad \text{and} \quad \{(r, \theta) \in \mathbb{R}^n; (1, \theta) \in \partial\Omega, r \in \mathbb{R}_+\}$$

in  $\mathbb{R}^n$  are denoted by  $C_n(\Omega)$  and  $S_n(\Omega)$ , respectively. If  $n=2$ , then  $C_2(\Omega)$  is an angular domain.

In the following, we put the strong assumption relative to  $\Omega$  on  $S^{n-1}$ : if  $n \geq 3$ ,  $\Omega$  is a  $C^{2,\alpha}$ -domain ( $0 < \alpha < 1$ ) on  $S^{n-1}$  surrounded by a finite number of mutually disjoint closed hypersurfaces (e.g. see [10, pp.88-89] for the definition of  $C^{2,\alpha}$ -domain).

Let  $G_{C_n(\Omega)}((r_1, \theta_1), (r_2, \theta_2))$  ( $(r_1, \theta_1), (r_2, \theta_2) \in C_n(\Omega)$ ) be the Green function of a cone  $C_n(\Omega)$  and let  $s_n$  denote the surface area  $2\pi^{n/2} \{\Gamma(n/2)\}^{-1}$  of  $S^{n-1}$ . The function

$$c_n^{-1} \frac{\partial}{\partial \nu} G_{C_n(\Omega)}(P, Q), \quad c_n = \begin{cases} 2\pi & (n=2) \\ (n-2)s_n & (n \geq 3) \end{cases}$$

of  $Q \in \partial C_n(\Omega)$  for any fixed  $P \in C_n(\Omega)$  is an ordinary Poisson kernel, where  $\frac{\partial}{\partial \nu}$  denotes the differentiation at  $Q$  along the inward normal into  $C_n(\Omega)$ . Let  $F(r, \theta)$  be a function on  $C_n(\Omega)$ . We put

$$(3) \quad \mu_0(F) = \lim_{r \rightarrow \infty} r^{-\alpha(\Omega, 1)} \int_{\Omega} F(r, \theta) f_1^{\Omega}(\theta) d\sigma_{\theta},$$

$$\text{and} \quad \eta_0(F) = \lim_{r \rightarrow \infty} r^{\beta(\Omega, 1)} \int_{\Omega} F(r, \theta) f_1^{\Omega}(\theta) d\sigma_{\theta},$$

if they exist, where  $d\sigma_{\theta}$  is the surface element on  $S^{n-1}$ .

**THEOREM  $Y_2$**  [18, Theorem 3 and Lemma 3]. Let  $g(Q) = g(t, \Xi)$  be a continuous function on  $S_n(\Omega)$  satisfying

$$(4) \quad \int_0^{+\infty} t^{-\alpha(\Omega, 1)-1} \left( \int_{\partial \Omega} |g(t, \Xi)| d\sigma_{\Xi} \right) dt < +\infty$$

$$\text{and} \quad \int_0^{+\infty} t^{\beta(\Omega, 1)-1} \left( \int_{\partial \Omega} |g(t, \Xi)| d\sigma_{\Xi} \right) dt < +\infty$$

(if  $n=2$  and  $\Omega = (\gamma, \delta)$ , then

$$\int_{\partial \Omega} |g(t, \Xi)| d\sigma_{\Xi} = |g(t, \gamma)| + |g(t, \delta)|).$$

Then the Poisson integral

$$H(C_n(\Omega); g)(P) = c_n^{-1} \int_{S_n(\Omega)} g(Q) \frac{\partial}{\partial \nu} G_{C_n(\Omega)}(P, Q) d\sigma_Q$$

is a solution of the classical Dirichlet problem on  $C_n(\Omega)$  with  $g$  such that

$$\mu_0(|H(C_n(\Omega);g)|) = 0 \quad \text{and} \quad \eta_0(|H(C_n(\Omega);g)|) = 0,$$

where  $d\sigma_\theta$  is the surface element on  $S^{n-1}$ . If  $h(P)$  is any solution of the classical Dirichlet problem on  $C_n(\Omega)$  with  $g$ , then all of the limits  $\mu_0(h)$ ,  $\eta_0(h)$  ( $-\infty < \mu_0(h)$ ,  $\eta_0(h) \leq +\infty$ ),  $\mu_0(|h|)$  and  $\eta_0(|h|)$  ( $0 \leq \mu_0(|h|)$ ,  $\eta_0(|h|) \leq +\infty$ ) exist, and if

$$\mu_0(|h|) < +\infty \quad \text{and} \quad \eta_0(|h|) < +\infty,$$

then

$$h(P) = H(C_n(\Omega), g)(P) + (\mu_0(h)r^{-\alpha(\Omega,1)} + \eta_0(h)r^{\beta(\Omega,1)})f_1^\Omega(\theta)$$

for any  $P=(r,\theta) \in C_n(\Omega)$ .

In this paper we shall show the existence of solutions of the Dirichlet problem on a cone (Theorems 1 and 2) and a type of uniqueness of them (Theorems 7 and 8) by introducing the conical generalized Poisson kernels and Poisson integrals, the special cases of which are  $H(T_n, \ell; g)$  and  $H(C_n(\Omega); g)(P)$ . They generalize Theorem  $Y_1$  to the conical case and Theorem  $Y_2$  to more unrestricted boundary function than (4). To prove the uniqueness, we shall give two results (Theorems 5 and 6) which are the conical version of Kuran's result [12, Theorem 10]. We also generalize the results of Finkelstein and Scheinberg [7] and Gardinar [8] to the conical case (Theorems 3 and 4). Finally a result of Yoshida [19, Theorem 3] will be generalized in the conical form (Theorem 9).

## 2. Results about the existence of solutions.

We denote the non-decreasing sequence of positive eigenvalues of (2) by  $\{\lambda(\Omega, k)\}_{k=1}^\infty$ . In this expression we write  $\lambda(\Omega, k)$  the same



number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding  $\lambda(\Omega, k)$  is denoted by  $f_k^\Omega(\theta)$ , the set of sequential eigenfunctions corresponding to the same value of  $\lambda(\Omega, k)$  in the sequence  $\{f_k^\Omega(\theta)\}_{k=1}^\infty$  makes an orthonormal basis for the eigenspace of the eigenvalue  $\lambda(\Omega, k)$ . We can also say that for each  $\Omega \subset S^{n-1}$  there is a sequence  $\{k_i\}$  of positive integers such that  $k_1=1$ ,  $\lambda(\Omega, k_i) < \lambda(\Omega, k_{i+1})$ ,

$$\lambda(\Omega, k_i) = \lambda(\Omega, k_i+1) = \lambda(\Omega, k_i+2) = \dots = \lambda(\Omega, k_{i+1}-1) \quad (i=1, 2, 3, \dots)$$

and  $\{f_{k_i}^\Omega, f_{k_i+1}^\Omega, \dots, f_{k_{i+1}-1}^\Omega\}$  is an orthonormal basis for the eigenspace of the eigenvalue  $\lambda(\Omega, k_i)$  ( $i=1, 2, 3, \dots$ ). It is well known that  $k_2=2$  and  $f_1^\Omega(\theta) > 0$  for any  $\theta \in \Omega$  (see Courant and Hilbert [5, p.451 and p.458]). With respect to  $\{k_i\}$ , the following Remark 1 shows that even in the case  $\Omega = S_+^{n-1}$  ( $n=2, 3, 4, \dots$ ), not only the simplest case  $k_i=i$  ( $i=1, 2, 3, \dots$ ) but also other complex cases can appear.

If we note that  $\Omega$  is an  $(n-1)$ -dimensional compact Riemannian manifold with its boundary to be sufficiently regular, we know that

$$\lambda(\Omega, k) \sim A(\Omega, n) k^{2/(n-1)} \quad (k \rightarrow +\infty)$$

(e.g. see Cheng and Li [4]) and

$$\sum_{\lambda(\Omega, k) \leq x} \{f_k^\Omega(\theta)\}^2 \sim B(\Omega, n) x^{(n-1)/2} \quad (x \rightarrow +\infty)$$

uniformly with respect to  $\theta$  (e.g. Minakshisundaram and Pleijel [13], and also Essén and Lewis [6, p.120 and pp.126-128]), where  $A(\Omega, n)$  and  $B(\Omega, n)$  are both constants depending  $\Omega$  and  $n$ , respectively. Hence there exist two positive constants  $M_1, M_2$  such that

$$(5) \quad M k_1^{2/(n-1)} \leq \lambda(\Omega, k) \quad (k=1, 2, \dots)$$

and

$$|f_k^\Omega(\theta)| \leq M_2 k^{1/2} \quad (\theta \in \Omega, k=1,2,\dots)$$

If we denote two solutions of the equation

$$t^2 + (n-2)t - \lambda(\Omega, k) = 0$$

by  $\alpha(\Omega, k)$ ,  $-\beta(\Omega, k)$  ( $\alpha(\Omega, k), \beta(\Omega, k) > 0$ ), then we also have

$$\alpha(\Omega, k), \beta(\Omega, k) \geq M_3 k^{1/(n-1)} \quad (k=1,2,\dots),$$

from (5), where  $M_3$  is a positive constant independent of  $k$ . We

remark that both

$$r^{\alpha(\Omega, k)} f_k^\Omega(\theta) \quad \text{and} \quad r^{-\beta(\Omega, k)} f_k^\Omega(\theta) \quad (k=1,2,\dots)$$

are harmonic on  $C_n(\Omega)$  and vanish continuously on  $S_n(\Omega)$ . For a

domain  $\Omega$  and the sequence  $\{k_i\}$  mentioned above, by  $I(\Omega, k_\ell)$  we denote

the set of all positive integers less than  $k_\ell$  ( $\ell=1,2,3,\dots$ ). In

spite of the fact  $I(\Omega, k_1) = \emptyset$ , the summation over  $I(\Omega, k_1)$  of a function

$S(k)$  of a variable  $k$  will be used by promising

$$\sum_{k \in I(\Omega, k_1)} S(k) = 0.$$

REMARK 1. Suppose  $\Omega = S_+^{n-1}$  ( $n \geq 2$ ). Then

$$\begin{aligned} (6) \quad c_n^{-1} \frac{\partial}{\partial v} G_{T_n}((r, \theta), (t, \Xi)) \\ = 2s_n^{-1} \sum_{k=0}^{\infty} c_{k, n+2} r^{k+1} t^{-k-n} \cos \theta_1 L_{k, n+2}(\cos \gamma) \end{aligned}$$

for any  $(X, y) = (r, \theta) \in T_n$  and  $(Z, 0) = (t, \Xi) \in \partial T_n$  satisfying  $r < t$ , where

$c_{k, n+2} = \binom{k+n-1}{k}$ ,  $L_{k, n+2}$  is the  $(n+2)$ -dimensional Legendre polynomial

of degree  $k$  and  $\gamma$  is the angle between  $M = (X, 0)$  and  $N = (Z, 0)$  defined by

$$\cos \gamma = \frac{(M, N)}{|M||N|}$$

(see Armitage [1, Theorem E]). On the other hand, Remark 4 in

Section 4 applied to  $\Omega = S_+^{n-1}$  gives the Fourier series expansion of

the function

$$c_n^{-1} \frac{\partial}{\partial v} G_{T_n}((r, \theta), (t, \Xi)) \quad (r < t)$$

of  $\theta$  with respect to the sequence of eigenfunctions of (2).

Hence, in comparison with (2-4) we obtain

$$(7) \quad \alpha(S_+^{n-1}, k_i) = i, \\ \beta(S_+^{n-1}, k_i) = n+i-2 \quad (i=1, 2, 3, \dots; n=2, 3, 4, \dots).$$

Consider the simplest case  $n=2$  i.e.  $\Omega=S_+^1$ . For  $(r, \theta_1) \in T_2$  and  $(t, \Xi)=t \in \mathbb{R}$ , we see  $\cos \gamma = \frac{t}{|t|} \sin \theta_1$  and hence

$$k_i = i \quad (i=1, 2, 3, \dots) \\ f_k^\Omega(\theta_1) = \rho_k \cos \theta_1 L_{k-1,4}(\sin \theta_1) \quad (k=1, 2, \dots),$$

where

$$\rho_k = \left( \int_{-2^{-1}\pi}^{2^{-1}\pi} (\cos \theta_1 L_{k-1,4}(\sin \theta_1))^2 d\theta_1 \right)^{-1/2}.$$

Next, suppose  $n=3$  i.e.  $\Omega=S_+^2$ . Then for  $(r, \theta)=(X, y) \in T_3$ ,  $\theta=(\theta_1, \theta_2)$  and  $(t, \Xi) \in \partial T_3 = \mathbb{R}^2$ ,  $\Xi=(\frac{\pi}{2}, \xi_2)$ , we see

$$\cos \gamma = \sin \theta_1 \sin \theta_2 \sin \xi_2 + \sin \theta_1 \cos \theta_2 \cos \xi_2.$$

If we put

$$L_{0,5} = \Phi_{0,0} = 1$$

and

$$\begin{aligned} L_{k,5}(\sin \theta_1 \sin \theta_2 \sin \xi_2 + \sin \theta_1 \cos \theta_2 \cos \xi_2) \\ = \Phi_{k,0}(\theta_1, \theta_2) \cos^k \xi_2 + \Phi_{k,1}(\theta_1, \theta_2) \cos^{k-2} \xi_2 + \dots \\ + \Phi_{k, [2^{-1}k]}(\theta_1, \theta_2) \cos^{k-2[2^{-1}k]} \xi_2 \\ + \Psi_{k,0}(\theta_1, \theta_2) \cos^{k-1} \xi_2 \sin \xi_2 + \Psi_{k,1}(\theta_1, \theta_2) \cos^{k-3} \xi_2 \sin \xi_2 + \dots \\ + \Psi_{k, [(k-1)/2]}(\theta_1, \theta_2) \cos^{k-1-2[(k-1)/2]} \xi_2 \sin \xi_2 \end{aligned} \quad (k=1, 2, 3, \dots),$$

then

$$k_i = 1 + \frac{(i-1)i}{2} \quad (i=1, 2, 3, \dots)$$

and

$$f_1^\Omega(\theta) = (2ns_n^{-1})^{1/2} \cos \theta_1,$$

$$f_{k_i+j}^\Omega(\theta) = \begin{cases} \rho_{k_i+j} \Phi_{i-1,j}(\theta_1, \theta_2) \cos \theta_1, & (j=0, 1, \dots, [(i-1)/2]) \\ \rho_{k_i+j} \Psi_{i-1, j-[(i-1)/2]-1}(\theta_1, \theta_2) \cos \theta_1 & (j=[(i-1)/2]+1, \dots, [(i-1)/2]+[(i-2)/2]+1) \\ & (i=2, 3, 4, \dots), \end{cases}$$

where

$$\rho_{k_i+j} = \begin{cases} \left\{ \int_{S_+^2} \Phi_{i-1,j}^2(\theta_1, \theta_2) \cos^2 \theta_1 d\sigma_\theta \right\}^{-1/2} & (j=0, 1, \dots, [(i-1)/2]) \\ \left\{ \int_{S_+^2} \Psi_{i-1, j-[(i-1)/2]-1}^2(\theta_1, \theta_2) \cos^2 \theta_1 d\sigma_\theta \right\}^{-1/2} & (j=[(i-1)/2]+1, \dots, [(i-1)/2]+[(i-2)/2]+1) \\ & (i=2, 3, 4, \dots). \end{cases}$$

The Fourier coefficient

$$\int_{\Omega} F(\theta) f_k^\Omega(\theta) d\sigma_\theta$$

of a function  $F(\theta)$  on  $\Omega$  with respect to the orthonormal sequence  $\{f_k^\Omega(\theta)\}$  is denoted by  $c(F, k)$ , if it exists. Now we shall define generalized Poisson kernels of the conical type. For two non-negative integers  $\ell, m$  and two points  $P=(r, \theta) \in C_n(\Omega)$ ,  $Q=(t, \Xi) \in S_n(\Omega)$ , we put

$$(8) \quad \bar{V}(C_n(\Omega), \ell)(P, Q) = \sum_{k \in I(\Omega, k_{\ell+1})} 2^{\alpha(\Omega, k)+n-1} c((H_\Xi)_1, k) t^{-\alpha(\Omega, k)-n+1} r^{\alpha(\Omega, k)} f_k^\Omega(\theta)$$

and

$$\underline{V}(C_n(\Omega), m)(P, Q) = \sum_{k \in I(\Omega, k_{m+1})} \left(\frac{3}{2}\right)^{\beta(\Omega, k)} 2^{n-1} c((H_\Xi)_3, k) t^{\beta(\Omega, k)-n+1} r^{-\beta(\Omega, k)} f_k^\Omega(\theta),$$

where

$$(H_{\Xi})_r(\theta) = c_n^{-1} \frac{\partial}{\partial v} G_{C_n(\Omega)}((r, \theta), (2, \Xi)) \quad (r=1, 3).$$

We introduce two functions of  $P \in C_n(\Omega)$  and  $Q = (t, \Xi) \in S_n(\Omega)$

$$\bar{W}(C_n(\Omega), \ell)(P, Q) = \begin{cases} \bar{V}(C_n(\Omega), \ell)(P, Q) & (1 \leq t < +\infty) \\ 0 & (0 < t < 1) \end{cases}$$

and

$$\underline{W}(C_n(\Omega), m)(P, Q) = \begin{cases} \underline{V}(C_n(\Omega), m)(P, Q) & (0 < t < 1) \\ 0 & (1 \leq t < +\infty). \end{cases}$$

The generalized Poisson kernel  $K(C_n(\Omega), \ell, m)(P, Q)$  with respect to  $C_n(\Omega)$  is defined by

$$K(C_n(\Omega), \ell, m)(P, Q) = c_n^{-1} \frac{\partial}{\partial v} G_{C_n(\Omega)}(P, Q) - \bar{W}(C_n(\Omega), \ell)(P, Q) - \underline{W}(C_n(\Omega), m)(P, Q).$$

In fact

$$K(C_n(\Omega), \ell, 0)(P, Q) = c_n^{-1} \frac{\partial}{\partial v} G_{C_n(\Omega)}(P, Q) - \bar{W}(C_n(\Omega), \ell)(P, Q) \quad (\ell \geq 1)$$

and

$$K(C_n(\Omega), 0, 0)(P, Q) = c_n^{-1} \frac{\partial}{\partial v} G_{C_n(\Omega)}(P, Q).$$

REMARK 2. Put  $\Omega = S_+^{n-1}$  and  $r_2 = 1$  in Remark 3 of Section 4. Then from (7) we have

$$c_n^{-1} \frac{\partial}{\partial v} G_{T_n}((r, \theta), (t, \Xi)) = \sum_{i=0}^{\infty} 2^{n-1+i} r^i t^{1-n-i} \left( \sum_{k=k_i}^{k_{i+1}-1} c((H_{\Xi})_1, k) f_k^{\Omega}(\theta) \right)$$

for any  $(r, \theta) \in T_n$  and any  $(t, \Xi) \in \partial T_n$  ( $r < t$ ), which is (6). Hence we obtain

$$2^{n+i} \left( \sum_{k=k_{i+1}}^{k_{i+2}-1} c((H_{\Xi})_1, k) f_k^{\Omega}(\theta) \right) = 2s_n^{-1} c_{i, n+2} \cos \theta_1 L_{i, n+2}(\cos \gamma) \quad (i=0, 1, 2, \dots).$$

Since

$$\bar{V}(\mathbb{T}_n, \ell)(P, Q) = \sum_{i=0}^{\ell-1} 2^{n+i} r^{i+1} t^{-n-i} \left( \sum_{k=k_{i+1}}^{k_{i+2}-1} c((H_{\Xi})_1, k) f_k^{\Omega}(\theta) \right)$$

from (7), we finally have

$$\bar{V}(\mathbb{T}_n, \ell)(P, Q) = 2s_n^{-1} \sum_{i=0}^{\ell-1} c_{i, n+2} r^{i+1} t^{-i-n} \cos \theta_1 L_{i, n+2}(\cos \gamma).$$

This shows that our kernel  $K(\mathbb{T}_n, \ell, 0)(P, Q)$  ( $\ell \geq 1$ ) coincides with ones in Armitage [11], Siegel [15] and Yoshida [19].

Let  $F(P) = F(r, \theta)$  be a function on  $C_n(\Omega)$  and put

$$N(F)(r) = \int_{\Omega} F(r, \theta) f_1^{\Omega}(\theta) d\sigma_{\theta}.$$

For two non-negative integers  $p$  and  $q$  we write

$$\mu_p(F) = \lim_{r \rightarrow \infty} r^{-\alpha(\Omega, k_{p+1})} N(F)(r) \quad \text{and} \quad \eta_q(F) = \lim_{r \rightarrow 0} r^{\beta(\Omega, k_{q+1})} N(F)(r),$$

if they exist. Since  $k_1 = 1$ , we know that these with  $p = q = 0$  are consistent with (3).

The following theorem is a generalization of the first part of Theorem  $Y_2$  which is the case  $\ell = m = 0$  of Theorem 1.

**THEOREM 1.** *Let  $\ell, m$  be two non-negative integers and  $g(Q) = g(t, \Xi)$  be a continuous function on  $S_n(\Omega)$  satisfying (9) with  $\ell$  and (10) with  $m$ :*

$$(9) \quad \int_0^{+\infty} t^{-\alpha(\Omega, k_{\ell+1})-1} \left( \int_{\partial\Omega} |g(t, \Xi)| d\sigma_{\Xi} \right) dt < +\infty$$

and

$$(10) \quad \int_0^t \beta(\Omega, k_{m+1})-1 \left( \int_{\partial\Omega} |g(t, \Xi)| d\sigma_{\Xi} \right) dt < +\infty.$$

Then

$$H(C_n(\Omega), \ell, m; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), \ell, m)(P, Q) d\sigma_Q$$

is a solution of the classical Dirichlet problem on  $C_n(\Omega)$  with  $g$  satisfying

$$\mu_l(|H(C_n(\Omega), l, m; g)|) = \mu_m(|H(C_n(\Omega), l, m; g)|) = 0.$$

To emphasize that Theorem 1 is also a natural generalization of the first part of Theorem  $Y_1$ , the following Theorem 2 is more desirable than Theorem 1.

**THEOREM 2.** Let  $g(Q)=g(t, \Xi)$  be a continuous function on  $\partial C_n(\Omega)$  satisfying (9) with a non-negative integer  $l$ . Then

$$H(C_n(\Omega), l, 0; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), l, 0)(P, Q) d\sigma_Q$$

is a solution of the classical Dirichlet problem on  $C_n(\Omega)$  with  $g$  satisfying

$$\mu_l(|H(C_n(\Omega), l, 0; g)|) = 0.$$

By taking  $\Omega = S_+^{n-1}$ , we obtain from (7)

**COROLLARY 1** (Yoshida [19, Theorem 1]). Let  $g(X)$  be a continuous function on  $\partial T_n = \mathbb{R}^{n-1}$  satisfying (1) with a non-negative integer  $l$ . Then  $H(T_n, l, 0; g)(P)$  is a solution of the Dirichlet problem on  $T_n$  with  $g$  such that

$$\mu_l(|H(T_n, l, 0; g)|) = 0.$$

To solve the Dirichlet problem on  $C_n(\Omega)$  with any function  $g(Q)$ , we shall define other Poisson kernels. Let  $\phi(t)$  (resp.  $\psi(t)$ ) be a positive continuous function of  $t \geq 1$  (resp.  $0 < t \leq 1$ ) satisfying

$$\varphi(1) = 2^{-\alpha(\Omega, 1)} \quad (\text{resp. } \psi(1) = (\frac{3}{2})^{-\beta(\Omega, 1)}).$$

For a domain  $\Omega \subset \mathbb{S}^{n-1}$  and the sequence  $\{\alpha(\Omega, k_i)\}_{i=1}^{\infty}$  (resp.  $\{\beta(\Omega, k_i)\}_{i=1}^{\infty}$ ), denote the set

$$\begin{aligned} & \{t \geq 1; -\alpha(\Omega, k_i) = (\log 2)^{-1} (\log(t^{n-1} \varphi(t)))\} \\ & (\text{resp. } \{0 < t \leq 1; -\beta(\Omega, k_i) = (\log \frac{3}{2})^{-1} \log(t^{n-1} \psi(t))\}) \end{aligned}$$

by  $\bar{S}(\Omega, \varphi, i)$  (resp.  $\underline{S}(\Omega, \psi, i)$ ). Then  $1 \in \bar{S}(\Omega, \varphi, 1)$  (resp.  $1 \in \underline{S}(\Omega, \psi, 1)$ ).

When there is an integer  $N$  such that  $\bar{S}(\Omega, \varphi, N) \neq \emptyset$  and  $\bar{S}(\Omega, \varphi, N+1) = \emptyset$

(resp.  $\underline{S}(\Omega, \psi, N) \neq \emptyset$  and  $\underline{S}(\Omega, \psi, N+1) = \emptyset$ ), denote the set  $\{i; 1 \leq i \leq N\}$  of

integers by  $\bar{J}(\Omega, \varphi)$  (resp.  $\underline{J}(\Omega, \psi)$ ). Otherwise, denote the set of all

positive integers by  $\bar{J}(\Omega, \varphi)$  (resp.  $\underline{J}(\Omega, \psi)$ ). Let  $\bar{t}(i) = \bar{t}(\Omega, \varphi, i)$

(resp.  $\underline{t}(i) = \underline{t}(\Omega, \psi, i)$ ) be the minimum (resp. maximum) of elements  $t$

in  $\bar{S}(\Omega, \varphi, i)$  (resp.  $\underline{S}(\Omega, \psi, i)$ ) for each  $i \in \bar{J}(\Omega, \varphi)$  (resp.  $\underline{J}(\Omega, \psi)$ ). In

the former case, we put  $\bar{t}(N+1) = +\infty$  (resp.  $\underline{t}(N+1) = 0$ ). Then  $\bar{t}(1) = 1$

(resp.  $\underline{t}(1) = 1$ ).

We define  $\bar{W}(C_n(\Omega), \varphi)(P, Q)$  ( $P \in C_n(\Omega)$ ,  $Q = (t, \Xi) \in S_n(\Omega)$ ) by

$$\bar{W}(C_n(\Omega), \varphi)(P, Q) = \begin{cases} 0 & (0 < t < 1) \\ \bar{V}(C_n(\Omega), i)(P, Q) & (\bar{t}(i) \leq t < \bar{t}(i+1); i \in \bar{J}(\Omega, \varphi)). \end{cases}$$

We also define  $\underline{W}(C_n(\Omega), \psi)(P, Q)$  ( $P \in C_n(\Omega)$ ,  $Q = (t, \Xi) \in S_n(\Omega)$ ) by

$$\underline{W}(C_n(\Omega), \psi)(P, Q) = \begin{cases} 0 & (1 < t < +\infty) \\ \underline{V}(C_n(\Omega), i)(P, Q) & (\underline{t}(i+1) < t \leq \underline{t}(i); i \in \underline{J}(\Omega, \psi)). \end{cases}$$

The Poisson kernel  $K(C_n(\Omega), \varphi, \psi)(P, Q)$  and  $K(C_n(\Omega), \varphi)(P, Q)$  ( $P \in C_n(\Omega)$ ,

$Q \in S_n(\Omega)$ ) are defined by

$$\begin{aligned} & K(C_n(\Omega), \varphi, \psi)(P, Q) \\ & = c_n^{-1} \frac{\partial}{\partial \nu} G_{C_n(\Omega)}(P, Q) - \bar{W}(C_n(\Omega), \varphi)(P, Q) - \underline{W}(C_n(\Omega), \psi)(P, Q) \end{aligned}$$

and



$$K(C_n(\Omega), \varphi)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{C_n(\Omega)}(P, Q) - \bar{W}(C_n(\Omega), \varphi)(P, Q).$$

Now we have

**THEOREM 3.** *Let  $g(Q)$  be a continuous function on  $S_n(\Omega)$ . Then there are two positive continuous functions  $\varphi(t)$  of  $t \geq 1$  and  $\psi(t)$  of  $0 < t \leq 1$  such that*

$$H(C_n(\Omega), \varphi, \psi; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), \varphi, \psi)(P, Q) d\sigma_Q$$

*is a solution of the Dirichlet problem on  $C_n(\Omega)$  with  $g$ .*

We can also obtain

**THEOREM 4.** *Let  $g(Q)$  be a continuous function on  $\partial C_n(\Omega)$ . Then there is a positive continuous function  $\varphi(t)$  of  $t \geq 1$  such that*

$$H(C_n(\Omega), \varphi; g)(P) = \int_{S_n(\Omega)} g(Q) K(C_n(\Omega), \varphi)(P, Q) d\sigma_Q$$

*is a solution of the Dirichlet problem on  $C_n(\Omega)$  with  $g$ .*

If we take  $\Omega = S_+^{n-1}$  in Theorem 4, then we have

**COROLLARY 2** (Finkelstein and Scheinberg [7] and Gardinar [8]).

*Let  $g(Q)$  be a continuous function on  $\partial T_n$ . Then there is a positive continuous function  $\varphi(t)$  of  $t \geq 1$  such that*

$$H(T_n, \varphi; g)(P) = \int_{\partial T_n} g(Q) K(T_n, \varphi)(P, Q) d\sigma_Q$$

*is a solution of the Dirichlet problem on  $T_n$  with  $g$ .*

### 3. Results about a type of uniqueness of solutions.

The following result is just a generalization of Picard's theorem stating that if  $H$  is a positive harmonic function in the Euclidean space then  $H$  is a constant.

Let  $h(r, \theta)$  be harmonic on  $\mathbb{R}^d$  ( $d \geq 2$ ). If, for some positive  $t$ ,

$$r^{-t-1} \mathfrak{M}(h^+)(r) \rightarrow 0 \quad (r \rightarrow +\infty), \quad \mathfrak{M}(h^+)(r) = \int_{S^{d-1}} h^+(r, \theta) d\sigma_\theta,$$

then for some positive integer  $\ell$  less than  $t+1$

$$h(r, \theta) = C + \sum_{k=1}^{\ell} P_k(r, \theta) \quad ((r, \theta) \in \mathbb{R}^d),$$

where  $C$  is a constant and  $P_k(r, \theta) = r^k Y_k(\theta)$  is a homogeneous harmonic polynomial of order  $k$  ( $Y_k(\theta)$  is a spherical harmonic function of Laplace) (see e.g. Brelot [2, Appendix, §26]).

It is well known that the potential theory in  $\mathbb{R}^n$  is intimately related to the potential theory in  $\mathbb{R}^{n+2}$  and many results on harmonic functions in  $\mathbb{R}^n$  can easily be obtained by a passage to  $\mathbb{R}^{n+2}$ . By using this fact, Kuran proved the following theorem.

**THEOREM K** (Kuran [12, Theorem 10]). Let  $h(X, y)$  ( $=h(r, \theta)$ ) be a harmonic function on  $T_n$  such that  $h$  vanishes continuously on  $\partial T_n$ .

If, for some positive  $t$ ,

$$\lim_{r \rightarrow \infty} r^{-t-2} \mathfrak{D}(yh^+, r) = 0, \quad \mathfrak{D}(yh^+, r) = (s_r^+)^{-1} \int_{S_r^+} yh^+(r, \theta) dS_r^+,$$

where  $S_r^+ = \{(r, \theta) \in T_n; \theta \in S_+^{n-1}\}$ ,  $s_r^+$  is the surface area of the spherical part of  $S_r^+$  and  $dS_r^+$  is the surface element of  $S_r^+$ , then

$$h = y\pi$$

in  $T_n$ , where  $\pi$  is a polynomial of  $(x_1, x_2, \dots, x_{n-1}, y)$  in  $\mathbb{R}^n$  of degree

less than  $t$  and even with respect to the variable  $y$ .

Though his method is not applicable, we shall try to extend these results for functions defined on cones, one of which is the half-space. And we can obtain

**THEOREM 5.** Let  $p, q$  be two positive integers and  $h(r, \theta)$  be a harmonic function in  $C_n(\Omega)$  vanishing continuously on  $S_n(\Omega)$ . If  $h$  satisfies (11) with  $p$  and (12) with  $q$ :

$$(11) \quad \mu_p(h^+) = 0$$

and

$$(12) \quad \eta_q(h^+) = 0,$$

then

$$h(r, \theta) = \sum_{k \in I(\Omega, k_{p+1})} A_k(h) r^{\alpha(\Omega, k)} f_k^\Omega(\theta) + \sum_{k \in I(\Omega, k_{q+1})} B_k(h) r^{-\beta(\Omega, k)} f_k^\Omega(\theta),$$

for every  $(r, \theta) \in C_n(\Omega)$ , where  $A_k(h)$  ( $k=1, 2, \dots, k_{p+1}-1$ ) and  $B_k(h)$  ( $k=1, 2, \dots, k_{q+1}-1$ ) are all constants.

In comparison with Theorem K, the following type is preferred.

**THEOREM 6.** Let  $h(r, \theta)$  be a harmonic function in  $C_n(\Omega)$  vanishing continuously on  $\partial C_n(\Omega)$ . If  $h$  satisfies (11) with a positive integer  $p$ , then

$$h(r, \theta) = \sum_{k \in I(\Omega, k_{p+1})} A_k(h) r^{\alpha(\Omega, k)} f_k^\Omega(\theta)$$

for every  $(r, \theta) \in C_n(\Omega)$ , where  $A_k(h)$  ( $k=1, 2, \dots, k_{p+1}-1$ ) are all constants.

When we specialize  $\Omega = S_+^{n-1}$  i.e.  $C_n(\Omega) = T_n$ , from Theorem 6 and (7), we obtain the following corollary 3. The equality

$$\mathcal{D}(yh^+, r) = 2s_n^{-1} r N(h^+)(r)$$

shows that this is equal to Theorem K

**COROLLARY 3.** *Let  $h(r, \theta)$  be a harmonic function in  $T_n$  vanishing continuously on  $\partial T_n$ . If, for some positive  $t$ ,*

$$\lim_{r \rightarrow \infty} r^{-(t+1)} N(h^+)(r) = 0,$$

*then*

$$h(r, \theta) = y\pi,$$

*where  $\pi$  is a polynomial in  $R^n$  with degree less than  $t$  and even respect to the variable  $y$ .*

By using Theorem 5, we can prove the following Theorem 7.

**THEOREM 7.** *Let  $l, m$  be two non-negative integers and  $p, q$  be two positive integers satisfying  $p \geq l, q \geq m$ . Let  $g(t, \Xi)$  be a continuous function on  $S_n(\Omega)$  satisfying (9) with  $l$  and (10) with  $m$ . If  $h(r, \theta)$  is a solution of the Dirichlet problem on  $C_n(\Omega)$  with  $g$  satisfying*

$$(13) \quad \mu_p(h^+) = 0$$

*and*

$$\eta_q(h^+) = 0,$$

*then*

$$h(r, \theta) = H(C_n(\Omega), l, m; g)(P) + \sum_{k \in I(\Omega, k_{p+1})} A_k(h) r^{\alpha(\Omega, k)} f_k^\Omega(\theta) + \sum_{k \in I(\Omega, k_{q+1})} B_k(h) r^{-\beta(\Omega, k)} f_k^\Omega(\theta)$$

for every  $P=(r,\theta)\in C_n(\Omega)$ , where  $A_k(h)$  ( $k=1,2,\dots,k_{p+1}-1$ ) and  $B_k(h)$  ( $k=1,2,\dots,k_{q+1}-1$ ) are all constants.

If we take  $\ell=m=0$  and  $p=q=1$  in Theorem 7, then we have the following result containing the second part of Theorem  $Y_2$ .

**COROLLARY 4.** Let  $g(Q)$  be a continuous function on  $S_n(\Omega)$  satisfying (4). If  $h(r,\theta)$  is a solution of the Dirichlet problem on  $C_n(\Omega)$  with  $g$  satisfying

$$\mu_1(h^+) = \eta_1(h^+) = 0,$$

then

$$h(r,\theta) = H(C_n(\Omega);g)(P) + A_1(h)r^{\alpha(\Omega,1)}f_1^\Omega(\theta) + B_1(h)r^{-\beta(\Omega,1)}f_1^\Omega(\theta)$$

for every  $P=(r,\theta)\in C_n(\Omega)$ , where  $A_1(h)=\mu_0(h)$  and  $B_1(h)=\eta_0(h)$ .

By Theorem 6, we also have

**THEOREM 8.** Let  $\ell$  be a non-negative integer and  $p$  be a positive integer satisfying  $\ell \leq p$ . Let  $g(t,\Xi)$  be a continuous function on  $\partial C_n(\Omega)$  satisfying (9) with  $\ell$ . If  $h(r,\theta)$  is a solution of the Dirichlet problem on  $C_n(\Omega)$  with  $g$  satisfying (13) with  $p$ , then

$$h(r,\theta) = H(C_n(\Omega),\ell,0;g)(P) + \sum_{k \in I(\Omega,k_{p+1})} A_k(h)r^{\alpha(\Omega,k)}f_k^\Omega(\theta)$$

for every  $P=(r,\theta)\in C_n(\Omega)$ , where  $A_k(h)$  ( $k=1,2,\dots,k_{p+1}-1$ ) are all constants.

If we put  $\Omega=S_+^{n-1}$ ,  $\ell=p$  and  $p=p$  (resp.  $\ell=p-1$  and  $p=p$ ) ( $p$  is a positive integer) in Theorem 8, we obtain from Corollary 3

**COROLLARY 5** (Yoshida [19, Theorem 2 (resp. Corollary 2)]). Let  $\rho$  be a positive integer and  $g(X)$  be a continuous function on  $\partial T_n = \mathbb{R}^{n-1}$  satisfying (1) with  $\rho$  (resp. (1) with  $\rho-1$ ). If  $h(P)$  is a solution of the Dirichlet problem on  $T_n$  with  $g$  such that

$$\lim_{r \rightarrow \infty} r^{-(\rho+1)} N(h^+)(r) = 0,$$

then

$$h(P) = H(T_n, \rho, 0; g)(P) + yF(h)(P)$$

$$(\text{resp. } h(P) = H(T_n, \rho-1, 0; g)(P) + yF(h)(P)),$$

where  $F(h)(P)$  is a harmonic polynomial (of  $P = (x_1, x_2, \dots, x_{n-1}, y) \in \mathbb{R}^n$ ) of at most degree  $\rho-1$  vanishing on  $\partial T_n$  and even with respect to the variable  $y$ .

The following Theorem 9 also generalizes a result of Yoshida [19, Theorem 3].

**THEOREM 9.** If  $h(r, \theta)$  is a harmonic function on  $C_n(\Omega)$  and is continuous on  $\overline{C_n(\Omega)}$  such that the restriction  $h = h|_{\partial C_n(\Omega)}$  of  $h$  to  $\partial C_n(\Omega)$  satisfies

$$\int_t^{+\infty} t^{-\alpha(\Omega, k_{l+1})-1} \left( \int_{\partial \Omega} |h(t, \Xi)| d\sigma_{\Xi} \right) dt < +\infty$$

for some non-negative integer  $l$  and

$$\overline{\lim}_{r \rightarrow \infty} \frac{\log N(h^+)(r)}{\log r} < +\infty,$$

then for some positive integer  $p$

$$h(r, \theta) = H(C_n(\Omega), l, 0; h)(P) + \sum_{k \in I(\Omega, k_{p+1})} A_k(h) r^{\alpha(\Omega, k)} f_k^{\Omega}(\theta)$$

at every  $P=(r,\theta)\in C_n(\Omega)$ , where  $A_k(h)$  ( $k=1,2,\dots,k_{p+1}-1$ ) are all constants.

The Proof of all results in this part will be found in Yoshida and Miyamoto [20].

## PART 3. The cylindrical case

### 1. Introduction

There is another typical unbounded domain which is a cylinder

$$\Gamma_n(D) = D \times R$$

with a bounded domain  $D \subset R^{n-1}$ . The existence and the uniqueness of solutions of the Dirichlet problem on  $\Gamma_n(D)$  with a continuous function on  $\partial\Gamma_n(D)$  are worth being inquired. In this direction, Yoshida [18] proved the following Theorem A. To state it we need some preliminaries.

Consider the Dirichlet problem

$$(14) \quad \begin{aligned} (\Delta_{n-1} + \lambda)f &= 0 \quad \text{in } D \\ f &= 0 \quad \text{on } \partial D \end{aligned}$$

for a bounded domain  $D \subset R^{n-1}$  ( $n \geq 2$ ), where  $\Delta_1 = d^2/dx^2$ . Let  $\lambda(D, 1)$  be the least positive eigenvalue of (14) and  $f_1^D(X)$  be the normalized eigenfunction corresponding to  $\lambda(D, 1)$ . In order to make the subsequent consideration simpler, we put a strong assumption on  $D$  throughout the whole this paper: If  $n \geq 3$ , then  $D$  is a  $C^{2,\alpha}$ -domain ( $0 < \alpha < 1$ ) in  $R^{n-1}$  surrounded by a finite number of mutually disjoint closed hypersurfaces (for example, see Gilberg and Trudinger [9, pp.88-89] for the definition of  $C^{2,\alpha}$ -domain). Let  $G_{\Gamma_n(D)}(P_1, P_2)$  be the Green function of  $\Gamma_n(D)$  ( $P_1, P_2 \in \Gamma_n(D)$ ) and  $\partial G_{\Gamma_n(D)}(P, Q)/\partial \nu$  be the differentiation at  $Q \in \partial\Gamma_n(D)$  along the inward normal into  $\Gamma_n(D)$  ( $P \in \Gamma_n(D)$ ).

Given a function  $F(X, y)$  on  $\Gamma_n(D)$ , we denote the function of  $y$  defined by the integral

$$\int_D F(X, y) f_1^D(X) dX$$

by  $N(F)(y)$ , where  $dX$  denotes the  $(n-1)$ -dimensional volume element. We write

$$\mu_0(N(F)) = \lim_{y \rightarrow \infty} \exp(-\sqrt{\lambda(D, 1)y}) N(F)(y)$$

and

$$\eta_0(N(F)) = \lim_{y \rightarrow -\infty} \exp(\sqrt{\lambda(D, 1)y}) N(F)(y),$$

if they exist.

**Theorem A** (Yoshida [18, Theorem 6]). *Let  $g(Q)$  be a continuous function on  $\partial\Gamma_n(D)$  satisfying*

$$(15) \quad \int_{-\infty}^{\infty} \exp(-\sqrt{\lambda(D, 1)|y|}) \left( \int_{\partial D} |g(X, y)| d\sigma_X \right) dy < \infty,$$



where  $d\sigma_X$  is the surface area element of  $\partial D$  at  $X$  and if  $n = 2$  and  $D = (\gamma, \delta)$ , then

$$\int_{\partial D} |g(X, y)| d\sigma_X = |g(\gamma, y)| + |g(\delta, y)|.$$

Then the Poisson integral

$$PI_g(P) = c_n^{-1} \int_{\partial \Gamma_n(D)} g(Q) \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) d\sigma_Q$$

is a solution of the Dirichlet problem on  $\Gamma_n(D)$  with  $g$ , where

$$c_n = \begin{cases} 2\pi & (n = 2) \\ (n - 2)s_n & (n \geq 3) \end{cases} \quad (s_n \text{ is the surface area of the unit sphere } S^{n-1})$$

and  $d\sigma_Q$  is the surface area element on  $\partial \Gamma_n(D)$  at  $Q$ . Let  $h(P)$  be any solution of the Dirichlet problem on  $\Gamma_n(D)$  with  $g$ . Then all of the limits  $\mu_0(N(h)), \eta_0(N(h))$  ( $-\infty < \mu_0(N(h)), \eta_0(N(h)) \leq \infty$ ),  $\mu_0(N(|h|))$  and  $\eta_0(N(|h|))$  ( $0 \leq \mu_0(N(|h|)), \eta_0(N(|h|)) \leq \infty$ ) exist, and if

$$(16) \quad \mu_0(N(|h|)) < \infty \quad \text{and} \quad \eta_0(N(|h|)) < \infty,$$

then

$$h(P) = PI_g(P) + (\mu_0(N(h)) \exp(\sqrt{\lambda(D, 1)}y) + \eta_0(N(h)) \exp(-\sqrt{\lambda(D, 1)}y)) f_1^D(X)$$

for any  $P = (X, y) \in \Gamma_n(D)$ .

This Theorem A shows that under the conditions (15) and (16) the existence and a type of uniqueness of solutions for the Dirichlet problem on  $\Gamma_n(D)$  can be proved, respectively.

If  $n = 2$ , then  $\Gamma_n(D)$  is a strip. The strip  $\Gamma_2((0, \pi))$  with  $D = (0, \pi)$  is simply denoted by  $\Gamma_2$ . With respect to the Dirichlet problem on  $\Gamma_2$ , Widder obtained

**Theorem B** (Widder [13, Theorems 1 and 3]). *If  $g_i(t)$  ( $i = 1, 2$ ) is a continuous function on  $R$  satisfying*

$$\int_{-\infty}^{\infty} |g_i(t)| \exp(-|t|) dt < \infty,$$

then

$$H(\Gamma_2; g_1, g_2)(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(x, t - y) g_1(t) dt + \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\pi - x, t - y) g_2(t) dt$$

$$(P(x, y) = \frac{\sin x}{\cosh y - \cos x})$$

is a harmonic function in  $\Gamma_2$  and a continuous function on  $\overline{\Gamma_2}$  such that

$$H(\Gamma_2; g_1, g_2)(0, y) = g_1(y) \quad \text{and} \quad H(\Gamma_2; g_1, g_2)(\pi, y) = g_2(y) \quad (-\infty < y < \infty).$$

If  $h(x, y)$  is a harmonic function in  $\Gamma_2$  and a continuous function on  $\overline{\Gamma_2}$  such that

$$h(0, y) = g_1(y), \quad h(\pi, y) = g_2(y) \quad (-\infty < y < \infty)$$

and

$$\int_0^\pi |h(x, y)| dx = o(e^{|y|}) \quad (|y| \rightarrow \infty),$$

then

$$h(x, y) = H(\Gamma_2; g_1, g_2)(x, y)$$

on  $\overline{\Gamma_2}$ .

Though by a conformal mapping a strip is reduced to  $T_2$  which was treated in [20] as a special case, it may be interested to be independently treated as a special case of cylinders.

In this paper, the first parts of Theorems A and B will be extended by defining generalized Poisson integrals with continuous functions under more unrestricted conditions than (15) and (16) (Theorem 1 and Corollary 1). We shall also prove that for any continuous function  $g$  on  $\Gamma_n(D)$  there is a solution of the Dirichlet problem on  $\Gamma_n(D)$  with  $g$  (Theorem 2 and Corollary 2). The results (Theorem 3 and Corollary 3) which generalize the second parts of Theorems A and B will be connected with a type of uniqueness of solutions for the Dirichlet problem on  $\Gamma_n(D)$ .

## 2. Statements of our results

We denote the non-decreasing sequence of positive eigenvalues of (15) by  $\{\lambda(D, k)\}_{k=1}^\infty$ . In this expression we write  $\lambda(D, k)$  the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding  $\lambda(D, k)$  is denoted by  $f_k^D$ , the set of sequential eigenfunctions corresponding to the same value of  $\lambda(D, k)$  in the sequence  $\{f_k^D\}_{k=1}^\infty$  makes an orthonormal basis for the eigenspace of the eigenvalue  $\lambda(D, k)$ . We can also say that for each  $D \subset R^{n-1}$  there is a sequence  $\{k_i\}$  of positive integers such that  $k_1 = 1, \lambda(D, k_i) < \lambda(D, k_{i+1})$

$$\lambda(D, k_i) = \lambda(D, k_i + 1) = \lambda(D, k_i + 2) = \dots = \lambda(D, k_{i+1} - 1)$$

and  $\{f_{k_i}^D, f_{k_i+1}^D, \dots, f_{k_{i+1}-1}^D\}$  is an orthonormal basis for the eigenspace of the eigenvalue  $\lambda(D, k_i)$  ( $i = 1, 2, 3, \dots$ ). It is well known that  $k_2 = 2$  and  $f_1^D(X) > 0$  for any  $X \in D$  (See Courant and Hilbert [5, p.451 and p.458]). With respect to  $\{k_i\}$ , the following Example (2) shows that even in the case where  $D$  is an open disk in  $R^2$ , not the simplest case  $k_i = i$  ( $i = 1, 2, 3, \dots$ ), but complex case can appear.

When  $D$  has sufficiently smooth boundary, we know that

$$\lambda(D, k) \sim A(D, n)k^{2/(n-1)} \quad (k \rightarrow \infty)$$

and

$$\sum_{\lambda(D, k) \leq x} \{f_k^D(X)\}^2 \sim B(D, n)x^{(n-1)/2} \quad (x \rightarrow \infty)$$

uniformly with respect to  $X \in D$ , where  $A(D, n)$  and  $B(D, n)$  are both constants depending on  $D$  and  $n$  (e.g. see Weyl [7] and Carleman [3]). Hence there exist two positive constants  $M_1, M_2$  such that

$$M_1 k^{2/(n-1)} \leq \lambda(D, k) \quad (k = 1, 2, \dots)$$

and

$$|f_k^D(X)| \leq M_2 k^{1/2} \quad (X \in D, k = 1, 2, \dots).$$

We remark that both

$$\exp(\sqrt{\lambda(D, k)y})f_k^D(X) \quad \text{and} \quad \exp(-\sqrt{\lambda(D, k)y})f_k^D(X) \quad (k = 1, 2, \dots)$$

are harmonic on  $\Gamma_n(D)$  and vanish continuously on  $\partial\Gamma_n(D)$ .

For a domain  $D$  and the sequence  $\{k_i\}$  mentioned above, by  $I(D, k_j)$  we denote the set of all positive integers less than  $k_j$  ( $j = 1, 2, 3, \dots$ ). In spite of the fact  $I(D, k_1) = \emptyset$ , the summation over  $I(D, k_1)$  of a function  $S(j)$  of a variable  $j$  will be used by promising  $\sum_{k \in I(D, k_1)} S(k) = 0$ .

**Examples.** (1) Let  $D = (0, \pi)$ . Then (14) is reduced to find solutions  $f(x)$  ( $0 \leq x \leq \pi$ ) such that

$$\frac{d^2 f(x)}{dx^2} + \lambda f(x) = 0 \quad (0 < x < \pi)$$

and

$$f(0) = f(\pi) = 0$$

It is easy to see that  $k_i = i$ ,  $\lambda(D, k) = k^2$  and  $f_k^D(x) = \sqrt{\frac{2}{\pi}} \sin kx$  ( $k = 1, 2, 3, \dots$ ).

(2) Let  $D = \{(x, y) \in R^2; x^2 + y^2 < 1\}$ . Let  $\{\alpha_{n,m}\}_{m=1}^\infty$  be the increasing sequence of positive real numbers  $\alpha_{n,m}$  such that

$$J_n(\alpha_{n,m}) = 0 \quad (n = 0, 1, 2, \dots),$$

where  $J_n(z)$  is the Bessel function of order  $n$ . If the spherical coordinates  $x = r \cos \theta, y = r \sin \theta$  ( $0 \leq r < 1, 0 \leq \theta < 2\pi$ ) are introduced, then  $J_n(\alpha_{n,m}r) \cos n\theta$  and  $J_n(\alpha_{n,m}r) \sin n\theta$  ( $n \neq 0, m = 1, 2, 3, \dots$ ) are two eigenfunctions corresponding to the eigenvalue  $\lambda = \alpha_{n,m}^2$  (see Courant and Hilbert [5]). Since we do not know how the zeros of Bessel functions distribute, we cannot explicitly determine the sequence  $\{k_i\}$  with respect this  $D$ .

The Fourier coefficient

$$\int_D F(X) f_k^D(X) dX$$

of a function  $F(X)$  on  $D$  with respect to the orthonormal sequence  $\{f_k^D(X)\}$  is denoted by  $c(F, k)$ , if it exists. Now we shall define generalized Poisson kernels. Let  $l$  and  $m$  be two non-negative integers. For two points  $P = (X, y) \in \Gamma_n(D)$ ,  $Q = (X^*, y^*) \in \partial\Gamma_n(D)$ , we put

$$\bar{V}(\Gamma_n(D), l)(P, Q)$$

$$= \sum_{k \in I(D, k_{l+1})} \exp(\sqrt{\lambda(D, k)}) c((H_{X^*})_1, k) f_k^D(X) \exp(\sqrt{\lambda(D, k)} y) \exp(-\sqrt{\lambda(D, k)} y^*),$$

and

$$\begin{aligned} & \underline{V}(\Gamma_n(D), m)(P, Q) \\ &= \sum_{k \in I(D, k_{m+1})} \exp(\sqrt{\lambda(D, k)}) c((H_{X^*})_1, k) f_k^D(X) \exp(-\sqrt{\lambda(D, k)} y) \exp(\sqrt{\lambda(D, k)} y^*), \end{aligned}$$

where

$$(H_{X^*})_1(X) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}((X, 1), (X^*, 0)).$$

We remark that  $\overline{V}(\Gamma_n(D), l)(P, Q)$  and  $\underline{V}(\Gamma_n(D), m)(P, Q)$  are two harmonic functions of  $P \in \Gamma_n(D)$  for any fixed  $Q \in \partial \Gamma_n(D)$ . We introduce two functions of  $P \in \Gamma_n(D)$  and  $Q = (X^*, y^*) \in \partial \Gamma_n(D)$

$$\overline{W}(\Gamma_n(D), l)(P, Q) = \begin{cases} \overline{V}(\Gamma_n(D), l)(P, Q) & (y^* \geq 0) \\ 0 & (y^* < 0) \end{cases}$$

and

$$\underline{W}(\Gamma_n(D), m)(P, Q) = \begin{cases} \underline{V}(\Gamma_n(D), m)(P, Q) & (y^* \leq 0) \\ 0 & (y^* > 0). \end{cases}$$

The Poisson kernel  $K(\Gamma_n(D), l, m)(P, Q)$  with respect to  $\Gamma_n(D)$  is defined by

$$K(\Gamma_n(D), l, m)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \overline{W}(\Gamma_n(D), l)(P, Q) - \underline{W}(\Gamma_n(D), m)(P, Q).$$

We note

$$K(\Gamma_n(D), 0, 0)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q).$$

Let  $p, q$  be two non-negative integers and  $I(y)$  be a function on  $R$ . The finite or infinite limits

$$\lim_{y \rightarrow \infty} \exp(-\sqrt{\lambda(D, k_{p+1})} y) I(y) \quad \text{and} \quad \lim_{y \rightarrow -\infty} \exp(\sqrt{\lambda(D, k_{q+1})} y) I(y)$$

are denoted by  $\mu_p(I)$  and  $\eta_q(I)$ , respectively, when they exist.

**Theorem 10.** *Let  $l, m$  be two non-negative integers and  $g(Q) = g(X^*, y^*)$  be a continuous function on  $S_n(D)$  satisfying*

$$(17) \quad \int_{-\infty}^{\infty} \exp(-\sqrt{\lambda(D, k_{l+1})} y^*) \left( \int_{\partial D} |g(X^*, y^*)| d\sigma_{X^*} \right) dy^* < \infty$$

and

$$\int_{-\infty}^{\infty} \exp(\sqrt{\lambda(D, k_{m+1})} y^*) \left( \int_{\partial D} |g(X^*, y^*)| d\sigma_{X^*} \right) dy^* < \infty.$$

Then

$$H(\Gamma_n(D), l, m, ; g)(P) = \int_{S_n(D)} g(Q) K(\Gamma_n(D), l, m)(P, Q) d\sigma_Q$$

is a solution of the Dirichlet problem on  $\Gamma_n(D)$  with  $g$  satisfying

$$\mu_l(N(|H(\Gamma_n(D), l, m; g)|)) = \eta_m(N(|H(\Gamma_n(D), l, m; g)|)) = 0.$$

If  $n = 2$  and  $D = (0, \pi)$ , then we immediately obtain the following Corollary 6 which generalizes Theorem B.

**Corollary 6.** *Let  $l, m$  be two non-negative integers and  $g_1(y^*), g_2(y^*)$  be two continuous functions on  $R$  satisfying*

$$(18) \quad \int_{-\infty}^{\infty} |g_i(y^*)| \exp(-(l+1)y^*) dy^* < \infty$$

and

$$\int_{-\infty}^{\infty} |g_i(y^*)| \exp(-(m+1)y^*) dy^* < \infty \quad (i = 1, 2).$$

Then

$$H(\Gamma_2, l, m; g_1, g_2)(x, y) = \int_{-\infty}^{\infty} g_1(y^*) K(\Gamma_2, l, m)((x, y), (0, y^*)) dy^* + \int_{-\infty}^{\infty} g_2(y^*) K(\Gamma_2, l, m)((x, y), (\pi, y^*)) dy^*$$

is a harmonic function in  $\Gamma_2$  and a continuous function on  $\overline{\Gamma_2}$  such that

$$H(\Gamma_2, l, m; g_1, g_2)(0, y^*) = g_1(y^*)$$

and

$$H(\Gamma_2, l, m; g_1, g_2)(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty)$$

To solve the Dirichlet problem on  $\Gamma_n(D)$  with any function  $g(Q)$  on  $\partial\Gamma_n(D)$ , we shall define another Poisson kernel. Let  $\varphi(t)$  be any positive continuous function of  $t \geq 0$  satisfying

$$\varphi(0) = \exp(-\sqrt{\lambda(D, 1)}).$$

For a domain  $D \subset R^{n-1}$  and the sequence  $\{\lambda(D, k_i)\}$ , denote the set

$$\{t \geq 0; \exp(-\sqrt{\lambda(D, k_i)}) = \varphi(t)\}$$

by  $S(D, \varphi, i)$ . Then  $0 \in S(D, \varphi, 1)$ . When there is an integer  $N$  such that  $S(D, \varphi, N) \neq \emptyset$  and  $S(D, \varphi, N+1) = \emptyset$ , denote the set  $\{i; 1 \leq i \leq N\}$  of integers by  $J(D, \varphi)$ . Otherwise, denote the set of all positive integers by  $J(D, \varphi)$ . Let  $t(i) = t(D, \varphi, i)$  be the minimum of elements  $t$  in  $S(D, \varphi, i)$  for each  $i \in J(D, \varphi)$ . In the former case, we put  $t(N+1) = \infty$ . Then  $t(1) = 0$ .

We define  $\overline{W}(\Gamma_n(D), \varphi)(P, Q)$  ( $P \in \Gamma_n(D)$ ,  $Q = (X^*, y^*) \in S_n(D)$ ) by

$$\overline{W}(\Gamma_n(D), \varphi)(P, Q) = \begin{cases} 0 & (y^* < 0) \\ \overline{V}(\Gamma_n(D), i)(P, Q) & (t(i) \leq y^* < t(i+1); i \in J(D, \varphi)). \end{cases}$$

We also define  $\underline{W}(\Gamma_n(D), \varphi)(P, Q)$  ( $P \in \Gamma_n(D)$ ,  $Q = (X^*, y^*) \in S_n(D)$ ) by

$$\underline{W}(\Gamma_n(D), \varphi)(P, Q) = \begin{cases} 0 & (y^* > 0) \\ \underline{V}(\Gamma_n(D), i)(P, Q) & (-t(i+1) < y^* \leq -t(i); i \in J(D, \varphi)). \end{cases}$$

The Poisson kernel  $K(\Gamma_n(D), \varphi)(P, Q)$  ( $P \in \Gamma_n(D)$ ,  $Q \in S_n(D)$ ) is defined by

$$K(\Gamma_n(D), \varphi)(P, Q) = c_n^{-1} \frac{\partial}{\partial \nu} G_{\Gamma_n(D)}(P, Q) - \overline{W}(\Gamma_n(D), \varphi)(P, Q) - \underline{W}(\Gamma_n(D), \varphi)(P, Q).$$

Now we have

**Theorem 11.** *Let  $g(Q)$  be a continuous function on  $\partial\Gamma_n(D)$ . Then there is a positive continuous function  $\varphi(t)$  of  $t \geq 0$  connected with  $g$  such that*

$$H(\Gamma_n(D), \varphi; g)(P) = \int_{\partial\Gamma_n(D)} g(Q) K(\Gamma_n(D), \varphi)(P, Q) d\sigma_Q$$

*is a solution of the Dirichlet problem on  $\Gamma_n(D)$  with  $g$ .*

If we take  $n = 2$  and  $D = (0, \pi)$  in Theorem 11, we obtain

**Corollary 7.** *Let  $g_1(y^*)$  and  $g_2(y^*)$  be two continuous functions on  $R$ . Then there is a positive continuous functions  $\varphi(t)$  of  $t \geq 0$  such that*

$$\begin{aligned} & H(\Gamma_2, \varphi; g_1, g_2)(x, y) \\ &= \int_{-\infty}^{\infty} g_1(y^*) K(\Gamma_2, \varphi)((x, y), (0, y^*)) dy^* + \int_{-\infty}^{\infty} g_2(y^*) K(\Gamma_2, \varphi)((x, y), (\pi, y^*)) dy^* \end{aligned}$$

*is a harmonic function in  $\Gamma_2$  and a continuous function on  $\overline{\Gamma_2}$  such that*

$$H(\Gamma_2, \varphi; g_1, g_2)(0, y^*) = g_1(y^*), \quad H(\Gamma_2, \varphi; g_1, g_2)(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty).$$

**Theorem 12.** *Let  $l, m$  be two non-negative integers and  $p, q$  be two positive integers satisfying  $p \geq l, q \geq m$ . Let  $g(X^*, y^*)$  be a continuous function on  $\partial\Gamma_n(D)$  satisfying (17). If  $h(X, y)$  is a solution of the Dirichlet problem on  $\Gamma_n(D)$  with  $g$  satisfying*

$$\mu_p(N(h^+)) = 0 \quad \text{and} \quad \eta_q(N(h^+)) = 0$$

*then*

$$\begin{aligned} & h(X, y) = H(\Gamma_n(D), l, m; g)(P) \\ &+ \sum_{k \in I(D, k_{p+1})} A_k(h) \exp(\sqrt{\lambda(D, k)} y) f_k^D(X) + \sum_{k \in I(D, k_{q+1})} B_k(h) \exp(-\sqrt{\lambda(D, k)} y) f_k^D(X) \end{aligned}$$

*for every  $P = (X, y) \in \Gamma_n(D)$ , where  $A_k(h)$  ( $k = 1, 2, \dots, k_{p+1} - 1$ ) and  $B_k(h)$  ( $k = 1, 2, \dots, k_{q+1} - 1$ ) are all constants.*

If we take  $n = 2$  and  $D = (0, \pi)$  in Theorem 12, then we have

**Corollary 8.** *Let  $l, m$  be two non-negative integers and  $p, q$  be two positive integers satisfying  $p \geq l, q \geq m$ . Let  $g_1(y^*), g_2(y^*)$  be two continuous function on  $R$  satisfying (18). If  $h(x, y)$  is a harmonic function in  $\Gamma_2$  and a continuous function on  $\overline{\Gamma_2}$  such that*

$$h(0, y^*) = g_1(y^*) \quad \text{and} \quad h(\pi, y^*) = g_2(y^*) \quad (-\infty < y^* < \infty),$$

and

$$\lim_{y \rightarrow \infty} \exp(-(p+1)y) \int_0^\pi h(x, y) \sin x dx = \lim_{y \rightarrow -\infty} \exp((q+1)y) \int_0^\pi h(x, y) \sin x dx = 0,$$

then

$$h(x, y) = H(\Gamma_2, l, m; g_1, g_2)(x, y) + \sum_{k=1}^p A_k(h) \exp(ky) \sin kx + \sum_{k=1}^q B_k(h) \exp(-ky) \sin kx$$

for every  $(x, y) \in \Gamma_2$ , where  $A_k(h)$  ( $k = 1, 2, \dots, p$ ) and  $B_k(h)$  ( $k = 1, 2, \dots, q$ ) are all constants.

The proof of all results in this part will be found in Miyamoto [14].

## References

- [1] Armitage, D.H.: Representations of harmonic functions in half-spaces. Proc. London Math. Soc. (3) 38 53-71 (1979)
- [2] Brelot, M.: Elements de la théorie classique du potentiel. Paris, 1965
- [3] Carleman, T.: Propriétés asymptotiques des fonctions fondamentales des membranes vibrantes. C. R. Skand. Math. Kongress 1934, 34-44
- [4] Cheng, S.Y., Li, P.: Heat kernel estimates and lower bound of eigenvalues. Comment. Math. Helvetici 56 327-338 (1981)
- [5] Courant, R., Hilbert, D.: Methods of mathematical physics. Vol.1 (Interscience Publishers, 1953)
- [6] Essén, M., Lewis, J.L.: The generalized Ahlfors-Heins theorems in certain  $d$ -dimensional cones. Math. Scand. 33 11-129 (1973)
- [7] Finkelstein, M., Scheinberg, S.: Kernels for solving problems of Dirichlet type in a half-plane. Advances in Math. 18 108-113 (1975)
- [8] Gardiner, S.J.: The Dirichlet and Neumann problems for harmonic functions in half-spaces. J. London Math. Soc.(2) 24 108-113 (1981)
- [9] Gilbarg, D., Trudinger, N.S.: Elliptic partial differential equations of second order. (Springer Verlag, 1977)
- [10] Hayman, W.K.: Subharmonic Functions. Vol.1, Academic, 1976
- [11] Helms, L.L.: Introduction to Potential Theory. (Wiley- Interscience, 1969)
- [12] Kuran, U.: Study of superharmonic functions in  $R^n \times (0, \infty)$  by a passage to  $R^{n+3}$ . Proc. London Math. Soc. 20 276-302 (1970)
- [13] Minakshisundaram, S., Pleijel, A.: Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. Canad. J. Math. 1 242-256 (1949)
- [14] Miyamoto, I.: A type of uniqueness of solutions for the Dirichlet problem on a cylinder. preprint
- [15] Siegel, D.: The Dirichlet problem in a half-spaces and a new Phragmén-Lindelöf principle, Maximum Principles and Eigenvalue Problems in Partial Differential Equations, e.d.P.W.Schaefer, Pitman, 1988
- [16] weyl, H.: Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen. Math. Ann. 71 441-471 (1911)
- [17] Widder, D.V.: Functions harmonic in a strip. Proc. Amer. Math. Soc. 12 67-72 (1961)



- [18] Yoshida, H.: Harmonic majorization of a subharmonic function on a cone or on a cylinder. *Pacific J. Math.* 148 369-395 (1991)
- [19] Yoshida, H.: A type of uniqueness for the Dirichlet problem on a half-spaces with continuous data. *Pacific J. Math.* (to appear)
- [20] Yoshida, H., Miyamoto, I.: Solutions of the Dirichlet problem on a cone with continuous data. preprint.